The Conjugacy Relation on Rank 2 Toeplitz Subshifts

Ruiwen Li (joint with Su Gao, Bo Peng and Yiming Sun)

Nankai University

Descriptive Dynamics and Combinatorics Seminar
March 18, 2025

Preliminary

Definition

A (topological dynamical) system is a pair (X, T) such that X is a compact metrizable space and T is an automorphism of X. For $A \subset X$, A is said to be invariant when $T(A) = A$. (X, T) is minimal if it has no nontrivial closed invariant subset (subsytem).

Definition

A (topological dynamical) system is a pair (X, T) such that X is a compact metrizable space and T is an automorphism of X. For $A \subset X$, A is said to be invariant when $T(A) = A$. (X, T) is minimal if it has no nontrivial closed invariant subset (subsytem).

Definition

For two systems (X, T) and (Y, S), if there exists a homeomorphism map f from X onto Y such that $fT = Sf$, then (X, T) is conjugate to (Y, S).

Definition

For a finite set A, let S be the left shift action on $A^{\mathbb{Z}}$, i.e.

$$S(x)(n) = x(n+1), \forall n \in \mathbb{Z},$$

then $(A^{\mathbb{Z}}, S)$ is a dynamical system, named Bernoulli shift. By a subshift we mean a subsystem of the Bernoulli shift.

Definition

For $x \in A^{\mathbb{Z}}$, let $\mathrm{Per}_p(x) = \{n \in \mathbb{Z} : \forall m \equiv n (\mathrm{mod}\, p)\, x(m) = x(n)\}$. The p-skeleton of x is $x \upharpoonright \mathrm{Per}_p(x)$. The periodic part of x is $\mathrm{Per}(x) = \bigcup_{n \in \mathbb{N}} \mathrm{Per}_n(x)$. The sequence x is a Toeplitz sequence if $\mathrm{Per}(x) = \mathbb{Z}$. A Toeplitz subshift is the orbit closure $\overline{\{S^n x : n \in \mathbb{Z}\}}$ of some Toeplitz sequence x.

Open Problem(Sabok–Tsankov)

What is the complexity of the conjugacy relation on Toeplitz subshifts under Borel reduction?

Fact

Both the set of minimal subshifts and the set of Toeplitz subshifts are G_δ subsets of $K(2^{\mathbb{Z}})$ under the Vietoris topology.

Fact

Both the set of minimal subshifts and the set of Toeplitz subshifts are G_δ subsets of $K(2^{\mathbb{Z}})$ under the Vietoris topology.

Theorem(Curtis–Hedlund–Lyndon)

For a conjugacy map f from a subshift (X, S) to another subshift (Y, S), there is $n \in \mathbb{N}$ and a map φ from A^{2n+1} to A, such that for every $x \in X$ and $k \in \mathbb{Z}$, $f(x)(k) = \varphi(x[k-n, k+n])$.

Fact

Both the set of minimal subshifts and the set of Toeplitz subshifts are G_δ subsets of $K(2^{\mathbb{Z}})$ under the Vietoris topology.

Theorem(Curtis–Hedlund–Lyndon)

For a conjugacy map f from a subshift (X, S) to another subshift (Y, S), there is $n \in \mathbb{N}$ and a map φ from A^{2n+1} to A, such that for every $x \in X$ and $k \in \mathbb{Z}$, $f(x)(k) = \varphi(x[k-n, k+n])$.

The map φ is called a block code of f.

Theorem(Thomas)
The conjugacy relation on Toeplitz subshifts is not smooth.

Theorem(Thomas)
The conjugacy relation on Toeplitz subshifts is not smooth.

Theorem(Sabok–Tsankov)
The conjugacy relation on Toeplitz subshifts with separated holes is amenable.

Theorem(Thomas)
The conjugacy relation on Toeplitz subshifts is not smooth.

Theorem(Sabok–Tsankov)
The conjugacy relation on Toeplitz subshifts with separated holes is amenable.

Theorem(Kaya)
The conjugacy relation on Toeplitz subshifts with growing blocks is hyperfinite.

If A, B are finite alphabets, a morphism $\phi : A^{<\omega} \to B^{<\omega}$ is a map satisfying that $\phi(\varnothing) = \varnothing$ and for all $u, v \in A^{<\omega}$, we have $\phi(uv) = \phi(u)\phi(v)$. We denote $A^{<\omega}$ by A^*.

If A, B are finite alphabets, a morphism $\phi : A^{<\omega} \to B^{<\omega}$ is a map satisfying that $\phi(\varnothing) = \varnothing$ and for all $u, v \in A^{<\omega}$, we have $\phi(uv) = \phi(u)\phi(v)$. We denote $A^{<\omega}$ by A^*.

A morphism $\phi : A^* \to B^*$ is primitive if for every $a \in A$ and $b \in B$, the element b occurs in $\phi(a)$; ϕ is proper if for every $a_1, a_2 \in A$, $\phi(a_1)$ and $\phi(a_2)$ begin with the same element in B and end with the same element in B; ϕ satisfies the constant length property if for every $a_1, a_2 \in A$, $|\phi(a_1)| = |\phi(a_2)|$.

If A, B are finite alphabets, a morphism $\phi : A^{<\omega} \to B^{<\omega}$ is a map satisfying that $\phi(\varnothing) = \varnothing$ and for all $u, v \in A^{<\omega}$, we have $\phi(uv) = \phi(u)\phi(v)$. We denote $A^{<\omega}$ by A^*.

A morphism $\phi : A^* \to B^*$ is primitive if for every $a \in A$ and $b \in B$, the element b occurs in $\phi(a)$; ϕ is proper if for every $a_1, a_2 \in A$, $\phi(a_1)$ and $\phi(a_2)$ begin with the same element in B and end with the same element in B; ϕ satisfies the constant length property if for every $a_1, a_2 \in A$, $|\phi(a_1)| = |\phi(a_2)|$.

Example For a morphism $\tau : \{0,1\}^* \to \{0,1\}^*$ such that $\tau(0) = 001$ and $\tau(1) = 01$, τ is primitive, proper but does not satisfy the constant length property.

A directive sequence is a sequence of morphisms

$$\tau = (\tau_n : A_{n+1}^* \to A_n^*)_{n\in\mathbb{N}},$$

τ is primitive (proper, constant length) if τ_n is primitive (proper, constant length) for every $n \in \mathbb{N}$.

$$A_0^* \leftarrow A_1^* \leftarrow A_2^* \leftarrow \cdots$$

A directive sequence is a sequence of morphisms

$$\tau = (\tau_n : A_{n+1}^* \to A_n^*)_{n \in \mathbb{N}},$$

τ is primitive (proper, constant length) if τ_n is primitive (proper, constant length) for every $n \in \mathbb{N}$.

$$A_0^* \leftarrow A_1^* \leftarrow A_2^* \leftarrow \cdots$$

For $n \in \mathbb{N}$, let

$$\tau_{[0,n]} = \tau_0 \circ \tau_1 \cdots \circ \tau_n,$$

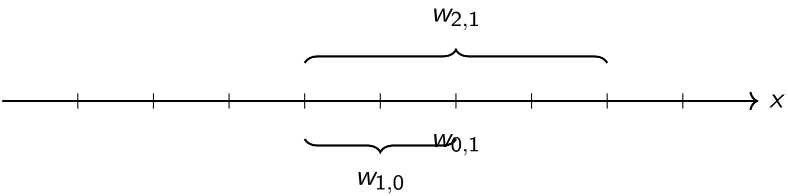
then the $\mathcal{S}$-adic subshift (X, S) generated by τ is the subshift that for every word $s \in A_0^*$, s is a subword of some point in X if and only if s is a subword of $\tau_{[0,n]}(a)$ for some $n \in \mathbb{N}$ and $a \in A_{n+1}$.

Example

Let $w_{0,0} = 0$, $w_{0,1} = 1$. For $n \in \mathbb{N}$, if $w_{n,0}$ and $w_{n,1}$ have been defined, let $w_{n+1,0} = w_{n,0}w_{n,0}w_{n,1}$ and $w_{n+1,1} = w_{n,0}w_{n,1}$, then $(w_{i,j})_{i\in\mathbb{N},1\leq j\leq 2}$ can be viewed as the directive sequence $(\tau_n : A_{n+1}^* \to A_n^*)_{n\in\mathbb{N}}$ where $A_n = \{0,1\}$ for every $n \in \mathbb{N}$, $\tau_n(0) = 001, \tau_n(1) = 01$ for every n. Actually $\tau_{[0,n]}(0) = w_{n,0}$ and $\tau_{[0,n]}(1) = w_{n,1}$.

Example

Let $w_{0,0} = 0$, $w_{0,1} = 1$. For $n \in \mathbb{N}$, if $w_{n,0}$ and $w_{n,1}$ have been defined, let $w_{n+1,0} = w_{n,0}w_{n,0}w_{n,1}$ and $w_{n+1,1} = w_{n,0}w_{n,1}$, then $(w_{i,j})_{i\in\mathbb{N},1\leq j\leq 2}$ can be viewed as the directive sequence $(\tau_n : A_{n+1}^* \to A_n^*)_{n\in\mathbb{N}}$ where $A_n = \{0,1\}$ for every $n \in \mathbb{N}$, $\tau_n(0) = 001, \tau_n(1) = 01$ for every n. Actually $\tau_{[0,n]}(0) = w_{n,0}$ and $\tau_{[0,n]}(1) = w_{n,1}$.



Fact

A subshift is minimal if and only if it is generated by a primitive directive sequence.

Fact

A subshift is minimal if and only if it is generated by a primitive directive sequence.

Fact

A subshift is Toeplitz if and only if it is generated by a proper, primitive and constant length directive sequence.

Definition

A directive sequence $(\tau_n : A_{n+1}^* \to A_n^*)_{n\in\mathbb{N}}$ is a rank 2 directive sequence if it is proper, primitive and $|A_n| = 2$ for every $n \geq 1$. Moreover, if $(\tau_n : A_{n+1}^* \to A_n^*)_{n\in\mathbb{N}}$ is constant length, then it is strong rank 2.

Definition

A directive sequence $(\tau_n : A_{n+1}^* \to A_n^*)_{n\in\mathbb{N}}$ is a rank 2 directive sequence if it is proper, primitive and $|A_n| = 2$ for every $n \geq 1$. Moreover, if $(\tau_n : A_{n+1}^* \to A_n^*)_{n\in\mathbb{N}}$ is constant length, then it is strong rank 2.

Definition

A (strong) rank 2 subshift is a subshift generated by a (strong) rank 2 directive sequence.

Theorem(Espinoza, Gao-L.)
The set of rank 2 subshifts is a Borel subset of $K(2^{\mathbb{Z}})$ under the Vietoris topology.

Theorem(Espinoza, Gao-L.)
The set of rank 2 subshifts is a Borel subset of $K(2^{\mathbb{Z}})$ under the Vietoris topology.

Theorem(Pavlov–Schmieding)
The set of rank 2 Toeplitz subshifts is a comeager subset of infinite minimal subshifts.

Theorem(Hjorth–Kechris)
For every countable Borel equivalence relation E on a Polish space X, there is a generic subset Y of X such that $E \restriction Y$ is hyperfinite.

Theorem(Hjorth–Kechris)
For every countable Borel equivalence relation E on a Polish space X, there is a generic subset Y of X such that $E \upharpoonright Y$ is hyperfinite.

Theorem(Gao–L.–Peng–Sun)
The conjugacy relation on the set of rank 2 Toeplitz subshifts is hyperfinite.

Rank 2 subshift: a rank 2 directive sequence

Toeplitz subshift: a proper, primitive, constant length directive sequence

Combination?

Theorem(Gao–L.–Peng–Sun)
There is a rank 2 but not strong rank 2 Toeplitz subshift.

Theorem(Gao–L.–Peng–Sun)
There is a rank 2 but not strong rank 2 Toeplitz subshift.

Theorem(Gao–L.–Peng–Sun)
The set of all strong rank 2 subshifts is a dense G_δ subset of infinite minimal subshifts.

Step 1: Toeplitz part

For $p \in \mathbb{N}^*$, p is an essential period of a sequence $x \in A^{\mathbb{Z}}$ if $\mathrm{Per}_p(x) \neq \mathrm{Per}_q(x)$ for every $q < p$. The scale of a Toeplitz sequence x is the supernatural number $\mathrm{u} = lcm(q_i)_{i \in \mathbb{N}}$ where q_i is an enumeration of the essential periods of x. For a supernatural number u, we can fix a canonical factorization $(p_n)_{n \in \mathbb{N}}$ such that $\mathrm{u} = lcm(p_n)_{n \in \mathbb{N}}$ and $p_n | p_{n+1}$ for every $n \in \mathbb{N}$.

For $p \in \mathbb{N}^*$, p is an essential period of a sequence $x \in A^{\mathbb{Z}}$ if $\mathrm{Per}_p(x) \neq \mathrm{Per}_q(x)$ for every $q < p$. The scale of a Toeplitz sequence x is the supernatural number $\mathrm{u} = lcm(q_i)_{i \in \mathbb{N}}$ where q_i is an enumeration of the essential periods of x. For a supernatural number u, we can fix a canonical factorization $(p_n)_{n \in \mathbb{N}}$ such that $\mathrm{u} = lcm(p_n)_{n \in \mathbb{N}}$ and $p_n | p_{n+1}$ for every $n \in \mathbb{N}$.

$$\forall i \exists j \; p_i | q_j \wedge \forall i \exists j \; q_i | p_j$$

For $p \in \mathbb{N}^*$, p is an essential period of a sequence $x \in A^{\mathbb{Z}}$ if $\mathrm{Per}_p(x) \neq \mathrm{Per}_q(x)$ for every $q < p$. The scale of a Toeplitz sequence x is the supernatural number $\mathrm{u} = lcm(q_i)_{i \in \mathbb{N}}$ where q_i is an enumeration of the essential periods of x. For a supernatural number u, we can fix a canonical factorization $(p_n)_{n \in \mathbb{N}}$ such that $\mathrm{u} = lcm(p_n)_{n \in \mathbb{N}}$ and $p_n | p_{n+1}$ for every $n \in \mathbb{N}$.

$$\forall i \exists j \; p_i | q_j \wedge \forall i \exists j \; q_i | p_j$$

Theorem(Williams)

For a Toeplitz subshift X and Toeplitz sequence $x \in X$, let $A^i_{p_n} = \{S^{kp_n+i}x : k \in \mathbb{Z}\}$, then $\overline{A^i_{p_n}}$ is a clopen subset of X, and $(\overline{A^i_{p_n}})_{0 \leq i \leq p_n - 1}$ is a partition of X.

For a Toeplitz sequence x, an interval $[m_1, m_2]$ on $\mathbb{Z}$ is a maximal p_n-periodic block of x if $[m_1, m_2] \subset \mathrm{Per}_{p_n}(x)$ but $m_1 - 1$ and $m_2 + 1$ don't belong to $\mathrm{Per}_{p_n}(x)$. For a maximal p_n-periodic block $[m_1, m_2]$, consider

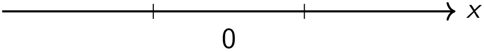
$$\overline{A_{p_n}^{\lfloor \frac{m_1+m_2}{2} \rfloor}} = \overline{\{S^{kp_n+\lfloor \frac{m_1+m_2}{2} \rfloor}(x) : k \in \mathbb{Z}\}}.$$

The collection of all such sets is a subcollection of $(\overline{A_{p_n}^i})_{0 \le i \le p_n - 1}$, denoted by $\gamma(X, p_n)$.

For a Toeplitz sequence x, an interval $[m_1, m_2]$ on $\mathbb{Z}$ is a maximal p_n-periodic block of x if $[m_1, m_2] \subset \mathrm{Per}_{p_n}(x)$ but $m_1 - 1$ and $m_2 + 1$ don't belong to $\mathrm{Per}_{p_n}(x)$. For a maximal p_n-periodic block $[m_1, m_2]$, consider

$$\overline{A_{p_n}^{\lfloor \frac{m_1+m_2}{2} \rfloor}} = \overline{\{S^{kp_n+\lfloor \frac{m_1+m_2}{2} \rfloor}(x) : k \in \mathbb{Z}\}}.$$

The collection of all such sets is a subcollection of $(\overline{A_{p_n}^i})_{0 \leq i \leq p_n - 1}$, denoted by $\gamma(X, p_n)$.



For $\phi \in Sym(A^p)$ and $x \in A^{\mathbb{Z}}$, let $\hat{\phi}(x) \in A^{\mathbb{Z}}$ be such that

$$\hat{\phi}(x)[kp, (k+1)p) = \phi(x[kp, (k+1)p)), \forall k \in \mathbb{Z}.$$

Two compact subsets K and L of $A^{\mathbb{Z}}$ are D_p equivalent if there is $\phi \in Sym(A^p)$ such that $\hat{\phi}(K) = L$.

For $\phi \in Sym(A^p)$ and $x \in A^{\mathbb{Z}}$, let $\hat{\phi}(x) \in A^{\mathbb{Z}}$ be such that

$$\hat{\phi}(x)[kp,(k+1)p) = \phi(x[kp,(k+1)p)), \forall k \in \mathbb{Z}.$$

Two compact subsets K and L of $A^{\mathbb{Z}}$ are D_p equivalent if there is $\phi \in Sym(A^p)$ such that $\hat{\phi}(K) = L$.

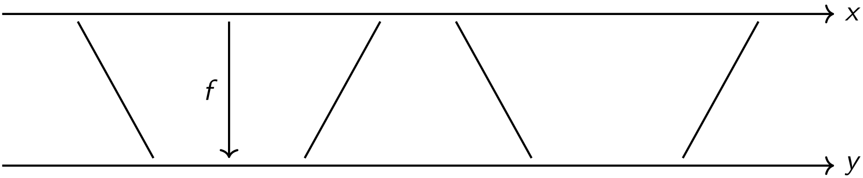
The relation D_p is a finite Borel equivalence relation.

For $\phi \in Sym(A^p)$ and $x \in A^{\mathbb{Z}}$, let $\hat{\phi}(x) \in A^{\mathbb{Z}}$ be such that

$$\hat{\phi}(x)[kp,(k+1)p) = \phi(x[kp,(k+1)p)), \forall k \in \mathbb{Z}.$$

Two compact subsets K and L of $A^{\mathbb{Z}}$ are D_p equivalent if there is $\phi \in Sym(A^p)$ such that $\hat{\phi}(K) = L$.

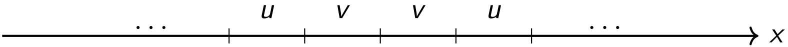
The relation D_p is a finite Borel equivalence relation.



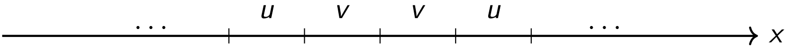
The map that sends a growing blocks Toeplitz subshift (X, S) to $(p_n, [\gamma(X, p_n)]_{D_{p_n}})_{n\in\mathbb{N}}$ is a Borel reduction witnessing the hyperfiniteness of the conjugacy relation on growing blocks Toeplitz subshifts

Step 2: Rank 2 Part

For a subshift (X, S) and $u, v \in A^*$, (u, v) is a recognizable pair in X if for every $x \in X$ there is a unique building of x from u and v. If $x[a, b) = u$ and a is a demonstrated position in the unique building of x, then a is an expected occurrence of u in x.



For a subshift (X, S) and $u, v \in A^*$, (u, v) is a recognizable pair in X if for every $x \in X$ there is a unique building of x from u and v. If $x[a, b) = u$ and a is a demonstrated position in the unique building of x, then a is an expected occurrence of u in x.

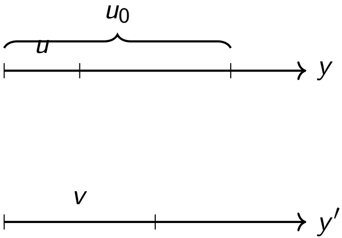


The set of $x \in X$ such that 0 is an expected occurrence of u in x is clopen.

Theorem(Espinoza)
Let (X, S) be a nontrivial rank 2 subshift, then for every $n \in \mathbb{N}$, there is a recognizable pair (u, v) in X such that the lengths of the common prefix and common suffix are greater than n.

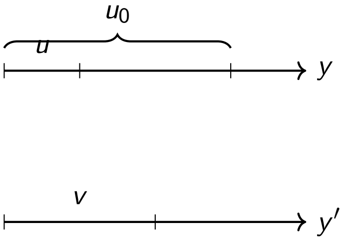
Lemma(Donoso-Durand-Maass-Petite)

Let u and v be different non-empty words satisfying that there is no $w \in A^*$ and $m, m' \in \mathbb{N}$ such that $u = w^m$ and $v = w^{m'}$. Then there exists $n < |u| + |v|$ and $u_0 \in A^n$ such that for any $y, y' \in 2^{\mathbb{Z}}$, if $y(0) \neq y'(0)$, then for all $0 \leq m < n$ we have $\tau(y)(m) = \tau(y')(m) = u_0(m)$ and $\tau(y)(n) \neq \tau(y')(n)$, where τ is the morphism maps 0 to u and 1 to v.



Lemma(Donoso-Durand-Maass-Petite)

Let u and v be different non-empty words satisfying that there is no $w \in A^*$ and $m, m' \in \mathbb{N}$ such that $u = w^m$ and $v = w^{m'}$. Then there exists $n < |u| + |v|$ and $u_0 \in A^n$ such that for any $y, y' \in 2^{\mathbb{Z}}$, if $y(0) \neq y'(0)$, then for all $0 \leq m < n$ we have $\tau(y)(m) = \tau(y')(m) = u_0(m)$ and $\tau(y)(n) \neq \tau(y')(n)$, where τ is the morphism maps 0 to u and 1 to v.



Here u_0 is called the distinguished prefix of u and v, similarly we can define distinguished suffix.

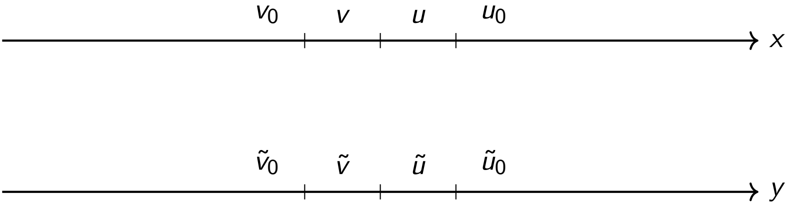
Step 3: Technical Lemma

For a Toeplitz subshift (X, S) and Toeplitz sequence $x \in X$, let $\underline{\chi(X, p_n)}$ be the set of all compact subsets of $A^{\mathbb{Z}}$ with the form $\overline{A_{p_n}^{\lfloor \frac{m_1+m_2}{2} \rfloor}}$ where $[m_1, m_2]$ be one of the **longest** maximal p_n periodic block of x.

For a Toeplitz subshift (X, S) and Toeplitz sequence $x \in X$, let $\underline{\chi(X, p_n)}$ be the set of all compact subsets of $A^{\mathbb{Z}}$ with the form $A_{p_n}^{\lfloor \frac{m_1+m_2}{2} \rfloor}$ where $[m_1, m_2]$ be one of the **longest** maximal p_n periodic block of x.

Lemma

For two rank 2 Toeplitz subshifts (X, S) and (Y, S), (X, S) and (Y, S) are conjugate if and only if they have the same scale and for $n \in \mathbb{N}$ large enough, $[\chi(X, p_n)]_{D_{p_n}} = [\chi(Y, p_n)]_{D_{p_n}}$.



Step 4: The reduction

The map that sends a rank 2 Toeplitz subshift (X, S) to $(p_n, [\chi(X, p_n)]_{D_{p_n}})_{n\in\mathbb{N}}$ is a Borel reduction witnessing the hyperfiniteness of the conjugacy relation.

Thank You!